UCD Senior Maths Enrichment: Graph Theory

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1 Basics

Definition 1. A **Graph** (G) is made of **vertices** (V) and **edges** (E). We write $G = (V, E)$.

Two vertices v_1 and v_2 are **adjacent** if they are connected by an edge. The **degree** of vertex v, written $d(v)$, is the number of edges of that vertex.

Remark 1. For this lecture, we are considering type of graph where:

1. The edges are undirected;
2. There can only be one edge between two vertices;
3. The edges are unweighted, i.e. "all edges are created the equal".

Lemma 1. *Let G be a graph with vertex set $V = \{v_1, v_2, ..., v_n\}$. Then $\sum_{k=1}^{n} d(v_k) = 2|E|$*

Proof. Every edge belongs to two vertices, so when we add up $\sum_{k=1}^{n} d(v_k)$, we have counted each edge twice.

More rigorously, we are counting number of pairs (v, e). By fixing v, we see this number is equal to $\sum_{k=1}^{n} d(v_k)$. But by fixing e, we see this number is equal to $2|E|$. ☐

Here we have used the idea from counting in two ways.

Definition 2 (Clique)**.** A **complete graph** on n vertices, denoted K_n, is the n-vertex graph with all possible edges.

Exercise 1. Prove there are $\binom{n}{2}$ edges in K_n.

Proof. There aren n vertices and an edge between any choice of two vertices. So $\binom{n}{2}$ edges. ☐

Definition 3. A path is a finite sequence of vertices $v_0, v_1, \ldots, v_n$ such that v_i and v_{i+1} are adjacent.

Definition 4. A cycle is a walk $v_0, v_1, \ldots, v_n$ such that $v_0 = v_n$.

Lemma 2. *Let G be a graph such that all of its vertices have degree 2. Prove that G is a union of pairwise disjoint cycles.*

Proof. Pick a vertex v_1, choose a neighbour and name is v_2. Suppose we have constructed $v_1, v_2, ..., v_n$, which are all different vertices. Then v_{n+1} has exactly one neighbour which is not v_{n-1}, let it be v_n. If $v_n = v_1$, terminate; If v_n is one of $v_2, v_3, ..., v_{n-2}$, contradiction. Otherwise continue. This process must terminate, which gives us a cycle. If this cycle contains all the vertices, we are done. If not, pick another vertex not in this cycle and continue. ☐

2 Connected Graphs

Definition 5. A graph G is **connected** if given any two vertices v_0 and v_1 of G there is a walk that starts in v_0 and ends in v_1. It is possible to decompose every graph into **connected components**.

Definition 6 (Tree)**.** If G is a connected graph without cycles, then G is a **tree**.

Exercise 2. Every tree with n vertices has exactly $n - 1$ edges.

Proof. We use induction. If $n = 1$, the assertion is clear. Suppose it is true for n and we want to prove it for $n+1$. We want to show that there is a vertex of degree 1.

In order for the graph to be connected, each vertex has degree at least 1. Now, we consider the longest possible path $v_0, v_1, ...v_k$ in the graph. If v_1 only makes an edge with v_2, we are done. Otherwise, we either have a cycle, or contradict maximality.

Thus v_1 must have degree 1, so we can delete this vertex to get a graph G' with only n vertices. Now, in order to use induction, we have to check that the new graph is also a tree.

G didn't have a cycle to begin with, so removing a vertex doesn't change that. G is connected, so if we take two vertices u_1 and u_2 in G', they are connected by path $u_1, w_0, w_1, \ldots, w_n, u_2$ in G. But none of $u_1, w_0, w_1, \ldots, w_n, u_2$ can be v_1, so G' is also connected. □

Here we used induction and the extremal principle, both very useful Olympiad techniques. Extremal principle means look at the features which are the maximum or minimum. Then you can get a contradiction by constructing something bigger or smaller.

3 Bipartite Graphs

Definition 7 (Bipartite Graph)**.** A graph is **bipartite** if the vertex set V can be partitioned into two sets X and Y such that all edges only goes from a vertex in X to a vertex in Y. In this case we write $G = (X, Y, E)$

Theorem 1 (König 1936)**.** *A graph G is bipartite iff all of its cycles have even lengths.*

Proof. **First**: we show that if G has a cycle of odd length, then it is not bipartite. Suppose that G has a cycle of length $2n+1$, and it was bipartite. That means the vertices of G can be divided into two parts, which we label 'left' or 'right'.

The vertices of the cycle could be split into the left part and the right part. One of the two parts must have at least $n+1$ vertices.

However, among any $n+1$ vertices of the cycle, there are two consecutive ones, so there must be an edge between either the left or the right part, contradicting the definition of bipartite graph.

Next, we suppose that G has no cycles of odd length. We show how to arrange the vertices into a bipartite graph.

Without the loss of generality, we can assume that G is connected.

Let 'left', 'right' be two empty sets. Choose v_0 and place it in 'left'. Given any other vertex v_1, take a path that begins in v_0 and ends in v_1. If the walk has even length we place v_1 in 'left', else in 'right'. Firstly this is a valid definition, because if there are two paths from v_0 to v_1, one odd and one even, we have an odd cycle, contradiction.

Also, there can be no edges between vertices on the same side, because if so, we also get an odd cycle. □

Definition 8 (Matching in bipartite graphs)**.** Suppose $G = (X, Y, E)$ is a bipartite graph with components X and Y, then a X-perfect matching is a subgraph $G' = (X, Y, E')$[1] such that:

1. Every vertex v in X has exactly one edge in E'.
2. Suppose v_1 and v_2 are two distinct vertices in X with edges e_1 and e_2 in E'. Also suppose e_1 connects $v_1 \in X$ with a vertex $u_1 \in Y$, e_2 connects $v_2 \in X$ with a vertex $u_2 \in Y$. Then $u_1 \neq u_2$.

In other words, there is an injective map from X to Y.

Definition 9. Given a subset V' of vertices of G, the **neighbourhood** of V' is defined to be the set of vertices adjacent to at least one vertex of V' and is denoted $N(V')$. Note that if $G = (X, Y, E)$ is a bipartite graph with components X, Y, and if $V' \subset X$, then $N(V') \subset Y$.

Theorem 2 (Hall's Marriage Theorem)**.** *Let $G = (X, Y, E)$ be a bipartite graph with components X and Y. Then G has a X-perfect matching iff $|S| \leq |N(S)|$ for all $S \subset X$.*

Remark 2. This makes intuitive sense, because if $|S| > |N(S)|$ for some subset S then we have no hope for finding a matching for S.

Exercise 3 (Kazakhstan 2003). We are given two square sheets of paper of area 2003. Each sheet is divided into 2003 polygons of area 1 (the divisions may be different). One sheet is placed on top of the other. Show that we can place 2003 pins such that each of the 4006 polygons is pierced.

[1] Here, a subgraph just means that $E' \subset E$.

Proof. Let $G = (X, Y, E)$ be a bipartite graph, where X represent the first square paper, Y represent the second square. Let X, Y each have 2003 vertices, representing the 2003 polygons. We draw an edge between a $u \in X$ and $v \in Y$ if the corresponding polygons overlap.

Then take $S \subset X$. The polygons in S cover an area of exactly $|S|$. $N(S)$ are all the polygons that overlaps with some polygon(s) in S, in other words, $N(S)$ covers all of S completely. Then we must have $|N(S)| \geq |S|$.

Thus by Hall's marriage theorem, we can find a matching, and we put a pin in every pair of polygons that are connected by this matching. □

4 Thinking algorithmically

Graph theory shows up in all sort of real life applications, and are espcially important in computer science.

Let's consider a weighted graph, which means a graph $G = (V, E)$, where each edge is labeled with a value.

This can be used to model, for example, a road map between a set V of cities. Each edge $e = (u, v)$ represent a road between cities u and v, and the value of that edge is the length of that road.

4.1 Krusgal's Algorithm

Now suppose that a hurricane Helene destroyed all of the roads. You need to repair some of these roads, but you want to get the job done as fast as possible. To do that, you want to find a subset of these roads to repair, such that:

1. The cities is still connected: you can get from any city to any other by the roads you choose to repair.
2. There are no cycles in this subset of roads.
3. The sum of the distances of these roads are the minimum.

Problem: Find the minimum weighted spanning tree with the shortest total length.

Intuition: To get the minimum possible, we probably want the shortest paths possible. So let's try to choose the shortest paths one by one. But the problem is in this way we might make a cycle. So in each step we choose the shortest avaible road, unless a cycles forms. If so we move on to the next smallest.

Krusgal's Algorithm: First, Sort all the edges in ascending order of their weights. Next, forms sets of the vertices in the graph, one for each vertex, initially containing just that vertex. The algorithm proceeds as follows, until $|V| - 1$ edges has been added to the set of spanning tree edges:

1. The next shortest edge (not yet in spanning tree set) is examined. If two vertex end points of the edge are in different sets, then the edge is added to the set of spanning tree edges. A new set is formed from the union of two vertex sets, the two old sets are dropped.
2. If the two vertex endpoints of the edge are already in the same set, the edge is ignored. I.e. this chooses the next smallest edge, unless doing so would form a cycle.

Here we have used the greedy algorithm. Applying greedy algorithm directly doesn't always give the global optimum, but it gives useful insight, and sometimes can be modified to find the global optimum.

5 Hamilton Cycle: NP problem

Definition 10 (Hamiltonian Cycles)**.** A **Hamiltonian Cycles** is a cycle that visits each vertex exactly once.

Determining whether a graph has a Hamiltonian cycle is a classic **NP-problem**. The **P vs NP** problem is one of the biggest unsolved problems of mathematics, and has lots of huge implications.

6 Homework

1. Prove that every connected graph G has a tree T that uses all of its vertices. Such a tree is called a spanning tree.

2. Prove that in any connected graph G with at least three vertices, there are two vertices such that if we remove any or both of them, the graph remains connected.

3. Show that every tree (with at least two vertices) has at least two vertices of degree 1.

4. Suppose that a graph has at least as many edges as vertices. Show that it contains a cycle.

5. Show that a bipartite graph G with n vertices has at most $\frac{n^2}{4}$ edges.

6. Let $G = (X, Y; E)$ be a bipartite graph (the vertices are partitioned into sets X and Y) such that $d(x) \geq d(y)$ for all $x \in X$ and $y \in Y$. Then there is a complete matching from X to Y. (Hint: use Hall's marriage theorem)

7. Bonus: Prove that Krusgal's algoeithm is optimal. Read about Dijkstra's algorithm. Read about the **P** vs **NP** problem.